

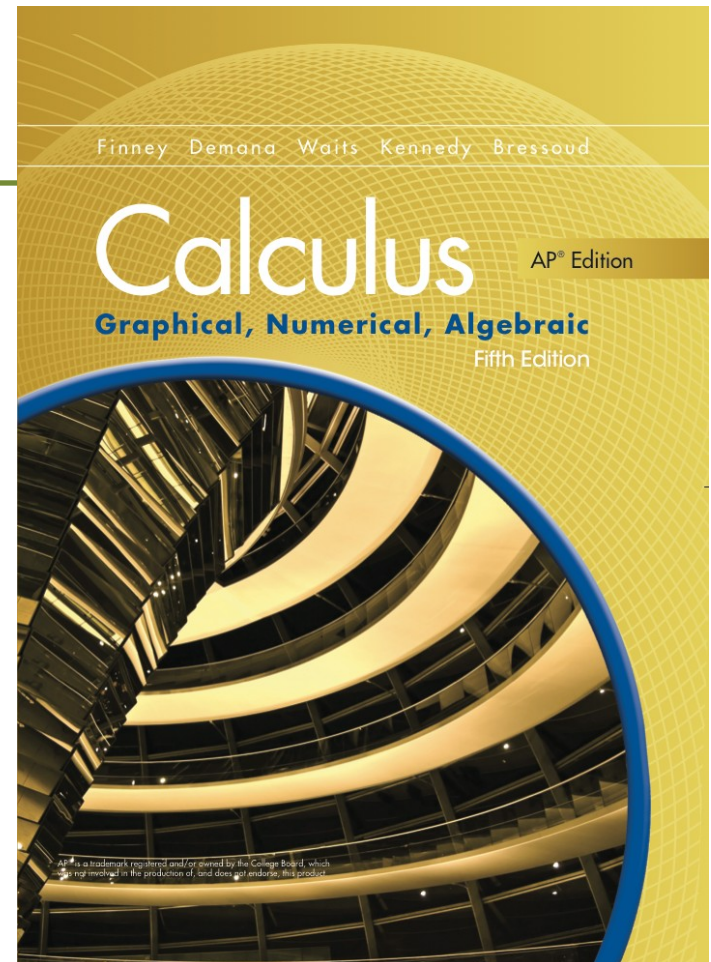
Chapter 10

Infinite Series



Section 10.5

Testing Convergence at Endpoints



Quick Review

Determine whether the improper integral converges or diverges.

1. $\int_1^{\infty} \frac{1}{x^{\frac{4}{3}}} dx$

2. $\int_1^{\infty} \frac{x^2}{x^3 + 1} dx$

3. $\int_1^{\infty} \frac{\ln x}{x} dx$

4. $\int_1^{\infty} \frac{1 + \cos x}{x^2} dx$

Quick Review

Determine whether the function is both positive and decreasing on some interval (N, ∞) .

5. $f(x) = 3/x$

6. $f(x) = \frac{7x}{x^2 - 8}$

7. $f(x) = \frac{3 + x^2}{3 - x^2}$

8. $f(x) = \frac{\sin x}{x}$

Quick Review Solutions

Determine whether the improper integral converges or diverges.

1. $\int_1^{\infty} \frac{1}{x^{\frac{3}{4}}} dx$ converges

2. $\int_1^{\infty} \frac{x^2}{x^3 + 1} dx$ diverges

3. $\int_1^{\infty} \frac{\ln x}{x} dx$ diverges

4. $\int_1^{\infty} \frac{1 + \cos x}{x^2} dx$ converges

Quick Review Solutions

Determine whether the function is both positive and decreasing on some interval (N, ∞) .

5. $f(x) = 3/x$ Yes

6. $f(x) = \frac{7x}{x^2 - 8}$ Yes

7. $f(x) = \frac{3 + x^2}{3 - x^2}$ No

8. $f(x) = \frac{\sin x}{x}$ No

What you'll learn about

- The Integral Test
- Harmonic series and p -series
- The Limit Comparison Test
- The Alternating Series Test and
- Leibniz's Theorem
- The strange implication of conditional convergence
- Finding intervals of convergence
- Why convergence of a series **for** f does not imply convergence **to** f

... and why

Additional tests for convergence of series are introduced in this section.

The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$ either both converge or both diverge.

Example Applying the Integral Test

Does $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converge?

The Integral Test applies because $f(x) = \frac{1}{x\sqrt{x}}$ is a continuous, positive, decreasing function of x for $x > 1$.

$$\begin{aligned}\int_1^{\infty} \frac{1}{x\sqrt{x}} dx &= \lim_{k \rightarrow \infty} \int_1^k x^{-3/2} dx = \lim_{k \rightarrow \infty} \left[-2x^{-1/2} \right]_1^k \\ &= \lim_{k \rightarrow \infty} \left(-\frac{2}{\sqrt{k}} + 2 \right) = 2\end{aligned}$$

Since the integral converges, so must the series.

Harmonic Series p -series

$\sum_{n=1}^{\infty} \left(\frac{1}{n^p} \right)$ is called a **p -series**, where p is a real constant.

A **harmonic series** is the p -series where $p=1$.

The Limit Comparison Test (LCT)

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N a positive integer).

1. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example Using the Limit Comparison Test

Determine whether the series converges or diverges.

$$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \dots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$

For n large, $\frac{2n+1}{(n+1)^2}$ behaves like $\frac{2n}{n^2} = \frac{2}{n}$, so compare it to the terms of $\sum (1/n)$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\cancel{(2n+1)} / \cancel{(n+1)^2}}{\cancel{1} / \cancel{n}} = \lim_{n \rightarrow \infty} \frac{2n+1}{(n+1)^2} \cdot \frac{n}{1} = 2 \quad (\text{Use l'Hôpital's Rule})$$

Since the limit is positive and $\sum (1/n)$ diverges, $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ also diverges.

The Alternating Series Test (Leibniz's Theorem)

The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$

converges if all three of the following conditions are satisfied:

1. each u_n is positive;
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N ;
3. $\lim_{n \rightarrow \infty} u_n = 0$.

The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies the conditions of the Alternating Series Test, then the truncation error for the n th partial sum is less than u_{n+1} and has the same sign as the first unused term.

Rearrangement of Absolutely Convergent Series

If $\sum a_n$ converges absolutely, and if $b_1, b_2, \dots, b_n, \dots$ is any rearrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Rearrangement of Conditionally Convergent Series

If $\sum a_n$ converges conditionally, then the terms can be rearranged to form a divergent series. The terms can also be rearranged to form a series that converges to any preassigned sum.

How to Test a Power Series for Convergence

1. Use the Ratio Test to find the values of x for which the series converges absolutely. Ordinarily, this is an open interval $a - R < x < a + R$. In some instances, the series converges for all values of x . In rare cases, the series converges only at $x = a$.
2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. The Ratio Test fails at these points. Use a comparison test, the Integral test, or the Alternating Series Test.
3. If the interval of absolute convergence is $a - R < x < a + R$, conclude that the series diverges for $|x - a| > R$, because for those values of x the n th term does not approach zero.

Example Finding Intervals of Convergence

For what values of x does the following series converge?

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n} = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \dots$$

Apply the Ratio Test to find the interval of absolute convergence, then check the endpoints if they exist.

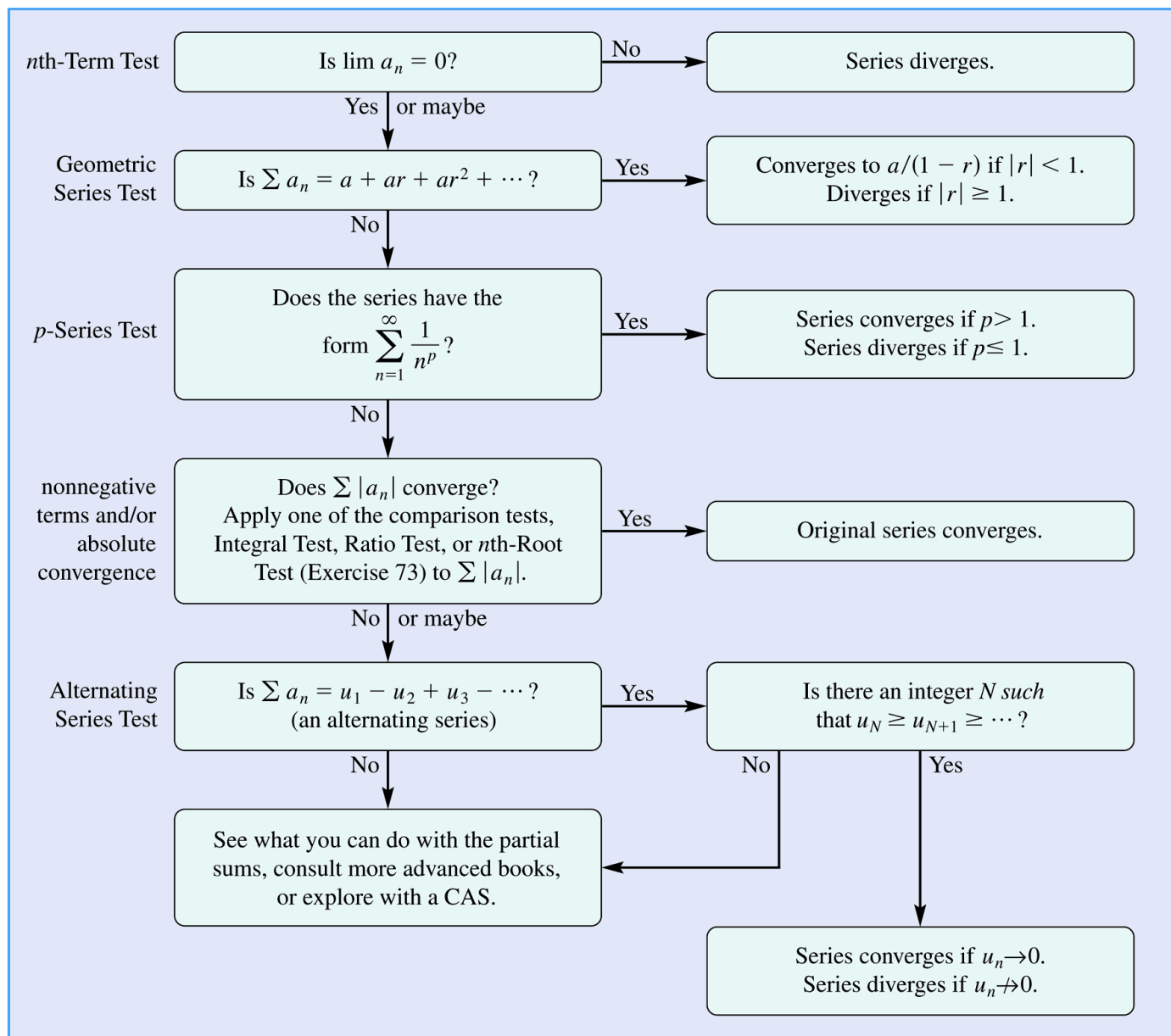
$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \frac{x^{2n+2}}{2n+2} \cdot \frac{2n}{x^{2n}} \\ &= \lim_{n \rightarrow \infty} \frac{2n}{2n+2} \cdot x^2 = x^2 \end{aligned}$$

The series converges absolutely for $x^2 < 1$ or on the interval $(-1, 1)$.

At $x=1$ and $x=-1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$. This converges by the

Alternating Series Test. The interval of convergence is $[-1, 1]$.

Procedure for Determining Convergence



Quick Quiz Sections 10.4 and 10.5

You may use a graphing calculator to solve the following problems.

1. Which of the following series converge?

I. $\sum_{n=0}^{\infty} \frac{2}{n^2 + 1}$ II. $\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n + 1}$ III. $\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{n}$

- (A) I only
- (B) II only
- (C) III only
- (D) II and III
- (E) I and II

Quick Quiz Sections 10.4 and 10.5

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(A) I only

(B) II only

(C) III only

(D) II and III

(E) I and II

Quick Quiz Sections 10.4 and 10.5

2. Which of the following is the sum of the telescoping series

$$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)}?$$

- (A) $1/3$
- (B) $1/2$
- (C) $3/5$
- (D) $2/3$
- (E) 1

Quick Quiz Sections 10.4 and 10.5

2. Which of the following is the sum of the telescoping series

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(A) $1/3$

(B) $1/2$

(C) $3/5$

(D) $2/3$

(E) 1

Quick Quiz Sections 10.4 and 10.5

3. Which of the following describes the behavior of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}?$$

I. converges II. diverges III. converges conditionally

(A) I only

(B) II only

(C) III only

(D) I and III only

(E) II and III only

Quick Quiz Sections 10.4 and 10.5

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$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}?$$

I. converges II. diverges III. converges conditionally

(A) I only

(B) II only

(C) III only

(D) I and III only

(E) II and III only

Chapter Test

Find **(a)** the radius of convergence for the series and **(b)** its interval of convergence. Then identify the values of x for which the series converges **(c)** absolutely and **(d)** conditionally.

1.
$$\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

2.
$$\sum_{n=0}^{\infty} \frac{(n+1)(2n+1)^n}{(2n+1)2^n}$$

3. The series is the value of the Maclaurin series of a function $f(x)$ at a particular point. What function and what point? What is the

sum of the series?
$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \dots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \dots$$

Chapter Test

4. Find a Maclaurin series for the function xe^{-x^2} .
5. Find the first four nonzero terms and the general term of the Taylor series generated by f at $x = a$.

$$f(x) = \frac{1}{3-x}, \quad a = 2$$

6. Find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(2n-3)(2n-1)}$.

Chapter Test

Determine if the series converges absolutely, converges conditionally, or diverges.

7.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

8.
$$\sum_{n=1}^{\infty} \frac{(-1)^n (n^2 + 1)}{2n^2 + n - 1}$$

9. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$ for all x for which the series converges.

- (a) Find the radius of convergence of this series.
- (b) Use the first three terms of this series to approximate $f(-1/3)$.
- (c) Estimate the error involved in the approximation in part (b).

Chapter Test

10. Let f be a function that has derivatives of all orders for all real numbers. Assume that $f(3) = 1$, $f'(3) = 4$, $f''(3) = 6$, and $f'''(3) = 12$.

(a) Write the third order Taylor polynomial for f at $x = 3$ and use it to approximate $f(3.2)$.

(b) Write the second order Taylor polynomial for f' at $x = 3$ and use it to approximate $f'(2.7)$.

(c) Does the linearization of f underestimate or overestimate the values of $f(x)$ near $x = 3$?

Chapter Test Solutions

Find **(a)** the radius of convergence for the series and **(b)** its interval of convergence. Then identify the values of x for which the series converges **(c)** absolutely and **(d)** conditionally.

1. $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$ (a) ∞ (b) all real numbers (c) all real numbers (d) none

2. $\sum_{n=0}^{\infty} \frac{(n+1)(2n+1)^n}{(2n+1)2^n}$ (a) 1 (b) $(-3/2, 1/2)$ (c) $(-3/2, 1/2)$ (d) none

3. The series is the value of the Maclaurin series of a function $f(x)$ at a particular point. What function and what point? What is the

sum of the series? $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \dots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \dots$

$f(x) = \sin x$ evaluated at $x = \pi$; sum = 0

Chapter Test Solutions

4. Find a Maclaurin series for the function xe^{-x^2} .

$$x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{n!} + \dots$$

5. Find the first four nonzero terms and the general term of the Taylor series generated by f at $x = a$.

$$f(x) = \frac{1}{3-x}, \quad a = 2$$

$$1 + (x-2) + (x-2)^2 + (x-2)^3 + \dots + (x-2)^n + \dots$$

6. Find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(2n-3)(2n-1)}$.

$$-1/6$$

Chapter Test Solutions

Determine if the series converges absolutely, converges conditionally, or diverges.

7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ converges conditionally

8. $\sum_{n=1}^{\infty} \frac{(-1)^n (n^2 + 1)}{2n^2 + n - 1}$ diverges

9. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$ for all x for which the series converges.

(a) Find the radius of convergence of this series. $1/e$

(b) Use the first three terms of this series to approximate $f(-1/3)$. $-5/18$

(c) Estimate the error involved in the approximation in part (b). $32/243 \approx 0.132$

Chapter Test Solutions

10. Let f be a function that has derivatives of all orders for all real numbers. Assume that $f(3) = 1$, $f'(3) = 4$, $f''(3) = 6$, and $f'''(3) = 12$.

(a) Write the third order Taylor polynomial for f at $x = 3$ and use it to approximate $f(3.2)$.

$$P_3(x) = 1 + 4(x - 3) + 3(x - 3)^2 + 2(x - 3)^3; f(3.2) \approx P_3(3.2) = 1.936$$

(b) Write the second order Taylor polynomial for f' at $x = 3$ and use it to approximate $f'(2.7)$.

$$P_2(x) = 4 + 6(x - 3) + 6(x - 3)^2; f'(2.7) \approx P_2(2.7) = 2.74$$

(c) Does the linearization of f **underestimate** or overestimate the values of $f(x)$ near $x = 3$?